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2005 J. Phys. A: Math. Gen. 38 4187

(http://iopscience.iop.org/0305-4470/38/19/009)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 4187–4208 doi:10.1088/0305-4470/38/19/009

A new matrix method for the Casimir operators of the Lie algebras $w\mathfrak{sp}(N,\mathbb{R})$ and $I\mathfrak{sp}(2N,\mathbb{R})$

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Received 13 November 2004, in final form 23 March 2005
Published 25 April 2005
Online at stacks.iop.org/JPhysA/38/4187

Abstract
A method is given to determine the Casimir operators of the perfect Lie algebras $w\mathfrak{sp}(N,\mathbb{R}) = \mathfrak{sp}(2N,\mathbb{R})\overrightarrow{\oplus}_{\Gamma_{\omega_1}\oplus\Gamma_0}\mathfrak{h}_N$ and the inhomogeneous Lie algebras $I\mathfrak{sp}(2N,\mathbb{R})$ in terms of polynomials associated with a parametrized $(2N+1)\times(2N+1)$-matrix. For the inhomogeneous symplectic algebras this matrix is shown to be associated to a faithful representation. We further analyse the invariants for the extended Schrödinger algebra $\widehat{S}(N)$ in $(N+1)$ dimensions, which arises naturally as a subalgebra of $w\mathfrak{sp}(N,\mathbb{R})$. The method is extended to other classes of Lie algebras, and some applications to the missing label problem are given.

PACS number: 02.20.Sv

1. Introduction

Symplectic Lie algebras constitute a quite interesting class of algebras for physical applications, due to their relation to some fundamental constructions. As is known, the Hamiltonian of the most general system of linear oscillators is given by

$$H = \alpha^{ij}p_ip_j + \beta^{ij}p_iq_j + \gamma^{ij}q_iq_j, \qquad 1 \leqslant i, j \leqslant N, \tag{1}$$

where $q = \{q_i\}$ and $p = \{p_i\}$ are the usual configuration and momentum space variables. It is straightforward to verify that the observables of degree 2 in p and q generate the real Lie algebra $\mathfrak{sp}(2N,\mathbb{R})$, while those of degree $\leqslant 1$ span the Heisenberg Lie algebra $\mathfrak{h}_N$. This constitutes evidence that both algebras, as well as the semidirect product $w\mathfrak{sp}(N,\mathbb{R})$ of $\mathfrak{sp}(2N,\mathbb{R})$ and $\mathfrak{h}_N$ can be of interest for the study of internal symmetry schemes of particles. In this context, the case $N = 3$ has been shown to play a distinguished role in the theory of nuclear collective motions [1, 2]. On the other hand, we find that the unitary algebra $\mathfrak{u}(N)$ (and therefore the $\mathfrak{su}(N)$ algebra) is naturally embedded into $\mathfrak{sp}(2N,\mathbb{R})$ as the centralizer of the element

$$H' = \tfrac{1}{2}(p^ip_i + q^iq_i). \tag{2}$$

Thus we also have a relation between symplectic groups and those applied to nuclear physics. Other important applications of the symplectic groups are, for example, the derivation of the dynamical noninvariance groups $SO(4, 2)$ for hydrogen-like atoms in three dimensions from the group $Sp(8, \mathbb{R})$, providing an additional approach to the classical method [3]. Another extremely important Lie algebra closely related to symplectic algebras, and worthy of analysis in this context, is the invariance algebra $\widehat{S}(N)$ of the Schrödinger equation in N-space and 1-time dimensions [4], which arises naturally as a subalgebra of the semidirect product $w\mathfrak{sp}(N, \mathbb{R})$.

It is therefore natural that the Casimir operators of these algebras are relevant for the different problems analysed, not only as an effective tool in the representation theory of these structures, but also for the obtention of quantum numbers and the labelling problems. In previous articles [5, 6] we have developed various methods to determine the Casimir operators of Lie algebras using determinants. These methods, applicable to Lie algebras having only a small number of invariants, are based either on the existence of certain extensions of the algebra or on the structure of the Levi part. The next logical step is trying to extend such methods for Lie algebras having an arbitrary number of invariants. The idea is to obtain a natural generalization of the matrix method introduced by Gel'fand in [7] for the simple Lie algebras. In this work we develop a method that enables us to obtain the Casimir operators of the Lie algebras $w\mathfrak{sp}(N, \mathbb{R})$ and $I\mathfrak{sp}(2N, \mathbb{R})$ from a determinant associated with a parametrized matrix obtained from an extension of the generic matrix of the standard representation of the symplectic algebra $\mathfrak{sp}(2N, \mathbb{R})$. This method provides the invariants directly, without the necessity of studying the corresponding enveloping algebras or taking contractions of Lie algebras, and is extremely easy to apply even for high values of N. For special cases we point out the relation of the matrix used and the existence of faithful representations of the algebra. As applications, it is shown that similar arguments hold for other Lie algebras such as the extended Schrödinger algebra $\widehat{S}(N)$, the Poincaré algebra and some of its contractions. We finally give an application to the missing label problem, where the missing label operators for certain subalgebra chains can be obtained directly by means of determinants.

The most widely used procedure to determine the (generalized) Casimir invariants of a Lie algebra $\mathfrak{g}$ is the analytical method, which turns out to be more practical than the traditional method of analysing the centre of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. This is particularly convenient in the study of completely integrable Hamiltonian systems, where Casimir operators in the classical sense do not have to exist, and where the transcendental invariant functions are not interpretable in terms of $\mathcal{U}(\mathfrak{g})$.

Given a basis $\{X_1, \ldots, X_n\}$ of the Lie algebra $\mathfrak{g}$ and the structure tensor $\left\{C_{ij}^k\right\}$, $\mathfrak{g}$ can be realized in the space $C^\infty(\mathfrak{g}^*)$ by means of differential operators:

$$\widehat{X}_i = -C_{ij}^k x_k \frac{\partial}{\partial x_j}, \tag{3}$$

where $[X_i, X_j] = C_{ij}^k X_k$ $(1 \leqslant i < j \leqslant n)$ and $\{x_1, \ldots, x_n\}$ is a dual basis of $\{X_1, \ldots, X_n\}$. In this context, an analytic function $F \in C^\infty(\mathfrak{g}^*)$ is called an invariant of $\mathfrak{g}$ if and only if it is a solution of the system of PDEs:

$$\{\widehat{X}_i F = 0, 1 \leqslant i \leqslant n\}. \tag{4}$$

Actually, for polynomials F in the commuting variables $\{x_1, \ldots, x_n\}$, $\widehat{X}_i F$ is nothing but the action of the generator X_i of $\mathfrak{g}$ on F. It is well known that polynomials invariant by this action, i.e., satisfying $\widehat{X}_i F = 0$, correspond to elements in the centre $Z(\mathfrak{U}(\mathfrak{g}))$ of the enveloping algebra of $\mathfrak{g}$ [8]. The explicit linear isomorphism between the set of polynomial solutions of (4) and $Z(\mathfrak{U}(\mathfrak{g}))$ is obtained from the symmetrization map $\mathcal{S}$. For any monomial $x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_p}$

of degree p define

$$\mathcal{S}(x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_p}) := \frac{1}{p!} \sum_{\sigma \in \mathcal{S}_p} X_{\alpha_{\sigma(1)}} X_{\alpha_{\sigma(2)}} \cdots X_{\alpha_{\sigma(p)}}, \tag{5}$$

where S_p is the symmetric group in p letters. Then the image of a polynomial $F(x_1, \ldots, x_n)$ is easily obtained by linear extension of (5). Nonpolynomial solutions of system (4) are usually called 'generalized Casimir invariants'. The cardinal $\mathcal{N}(\mathfrak{g})$ of a maximal set of functionally independent solutions (in terms of the brackets of the algebra $\mathfrak{g}$ over a given basis) is easily obtained from the classical criteria for PDEs:

$$\mathcal{N}(\mathfrak{g}) := \dim \mathfrak{g} - \operatorname{rank}\left(C_{ij}^k x_k\right)_{1 \leqslant i < j \leqslant \dim \mathfrak{g}}, \tag{6}$$

where $A(\mathfrak{g}) := \left(C_{ij}^k x_k\right)$ is the matrix which represents the commutator table of $\mathfrak{g}$ over the basis $\{X_1, \ldots, X_n\}$. Evidently this quantity constitutes an invariant of the algebra. We remark that $\mathcal{N}(\mathfrak{g})$ can also be obtained from the Maurer–Cartan equations of the Lie group [9], which is of interest in the context of the missing label problem and the classification of subalgebras, which will be discussed in section 7.

As mentioned above, real symplectic Lie algebras can easily be realized in terms of creation and annihilation operators [10]: consider the linear operators $a_i, a_j^\dagger$ $(i, j = 1, \ldots, N)$ satisfying the commutation relations

$$\left[a_i, a_j^\dagger\right] = \delta_{ij} \mathbb{I} \tag{7}$$

$$[a_i, a_j] = \left[a_i^\dagger, a_j^\dagger\right] = 0. \tag{8}$$

Considering the operators $\left\{a_i^\dagger a_j, a_i^\dagger a_j^\dagger, a_i a_j\right\}$, we generate the real symplectic Lie algebra $\mathfrak{sp}(2N, \mathbb{R})$. The brackets are easily obtained from (7) and (8). For practical purposes, we label the basis in the following form:

$$X_{i,j} = a_i^\dagger a_j, \qquad 1 \leqslant i, j \leqslant N \tag{9}$$

$$X_{-i,j} = a_i^\dagger a_j^\dagger \tag{10}$$

$$X_{i,-j} = a_i a_j. \tag{11}$$

The brackets of $\mathfrak{sp}(2N, \mathbb{R})$ can then be combined in a unique equation:

$$[X_{i,j}, X_{k,l}] = \delta_{jk} X_{il} - \delta_{il} X_{kj} + \varepsilon_i \varepsilon_j \delta_{j,-l} X_{k,-i} - \varepsilon_i \varepsilon_j \delta_{i,-k} X_{-j,l}, \tag{12}$$

where $-N \leqslant i, j, k, l \leqslant N$, $\varepsilon_i = \operatorname{sgn}(i)$ and

$$X_{i,j} + \varepsilon_i \varepsilon_j X_{-j,-i} = 0. \tag{13}$$

While this last basis is useful for the study of realizations of symplectic Lie algebras [11], the boson basis is more convenient for studying the semidirect products with Heisenberg algebras and their contractions [12–15]. In fact, the operators $a_i, a_i^\dagger$ transform as follows by the generators $\left\{a_i^\dagger a_j, a_i^\dagger a_j^\dagger, a_i a_j\right\}$ of $\mathfrak{sp}(2N, \mathbb{R})$:

$$\left[a_i^\dagger a_j, a_k^\dagger\right] = \delta_{jk} a_i^\dagger \tag{14}$$

$$\left[a_i^\dagger a_j, a_k\right] = -\delta_{ik} a_j \tag{15}$$

$$\left[a_i^\dagger a_j^\dagger, a_k\right] = -\delta_{jk} a_i^\dagger - \delta_{ik} a_j^\dagger \tag{16}$$

$$\left[a_i a_j, a_k^\dagger\right] = \delta_{ki} a_j + \delta_{kj} a_i. \tag{17}$$

With the labelling $P_i = a_i^\dagger$, $Q_i = a_i$ for $i = 1, \ldots, N$, we immediately see that (14)–(17) is nothing but the standard $2n$-dimensional representation Γ_{ω_1} of $\mathfrak{sp}(2N, \mathbb{R})$. Equations (7)–(8)

and (12)–(17) are the brackets for the semidirect product $w\mathfrak{sp}(N, \mathbb{R})$ of the symplectic algebra $\mathfrak{sp}(2N, \mathbb{R})$ with the $(2N + 1)$-dimensional Heisenberg–Weyl Lie algebra $\mathfrak{h}_N$ over the basis $\{X_{i,j}, P_k, Q_k, \mathbb{I}\}$ [12]. This type of construction for semidirect products is typical for the study of shift operator contractions and coherent state realisations of Lie algebras [12, 16].

Explicit expressions for the Casimir operators of semisimple Lie algebras have been proposed by many authors [17, 18], and even for nonsemisimple algebras there exist various procedures [5–7, 12, 19]. We recall in this section a quite economical method to determine the invariants of $w\mathfrak{sp}(2N, \mathbb{R})$ over the boson basis, and based on the classical matrix methods.

Proposition 1. *Let $N \geqslant 2$. Then the Casimir operators C_{2k} of $\mathfrak{sp}(2N, \mathbb{R})$ are given by the coefficients of the characteristic polynomial*

$$|A - T\,\mathrm{Id}_{2N}| = T^{2N} + \sum_{k=1}^{N} C_{2k} T^{2N-2k}, \tag{18}$$

where

$$A = \begin{pmatrix} x_{1,1} & \cdots & x_{1,N} & -x_{-1,1} & \cdots & -x_{-1,N} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{N,1} & \cdots & x_{N,N} & -x_{-1,N} & \cdots & -x_{-N,N} \\ x_{1,-1} & \cdots & x_{1,-N} & -x_{1,1} & \cdots & -x_{N,1} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{1,-N} & \cdots & x_{N,-N} & -x_{1,N} & \cdots & -x_{N,N} \end{pmatrix}. \tag{19}$$

Moreover $\deg C_{2k} = 2k$ *for* $k = 1, \ldots, N$.

The proof follows easily from the classical formulae of Perelomov and Popov [20], or using the trace method introduced by Gruber and O'Raifeartaigh in [18]. Observe that in fact the matrix A can be rewritten as

$$A = \sum_{i=1}^{N} x_{i,j} \Gamma_{\omega_1}(X_{i,j}), \tag{20}$$

where $\Gamma_{\omega_1}(X_{i,j})$ is the matrix corresponding to the generator $X_{i,j}$ by the standard representation Γ_{ω_1} of $\mathfrak{sp}(2N, \mathbb{R})$.

2. The Lie algebras $w\mathfrak{sp}(N, \mathbb{R})$

As we have seen, the Lie algebras $w\mathfrak{sp}(N, \mathbb{R}) = \mathfrak{sp}(2N, \mathbb{R}) \overrightarrow{\oplus}_{\Gamma_{\omega_1} \oplus \Gamma_0} \mathfrak{h}_N$ follow naturally from the boson realization of $\mathfrak{sp}(2N, \mathbb{R})$. This fact has important consequences for the applications of these algebras and their irreducible representations, such as the theory of nuclear collective motions [2].

Among the various methods to obtain the Casimir operators of the semidirect products $w\mathfrak{sp}(N, \mathbb{R})$, Quesne introduced in [12] a quite practical procedure, which has been shown recently to hold also for exceptional Lie algebras [13]. The main idea is to obtain a semisimple Lie algebra $\mathfrak{g}'$ isomorphic to $\mathfrak{sp}(2N, \mathbb{R})$ in the enveloping algebra $\mathfrak{U}$ of $w\mathfrak{sp}(N, \mathbb{R})$ such that its generators commute with the Heisenberg algebra. More generally, the Quesne theorem states that given a semidirect product $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{h}_N$ of a simple Lie algebra $\mathfrak{s}$ spanned by $\{X_1, \ldots, X_r\}$ and a Heisenberg algebra spanned by $\{Y_1, \ldots, Y_{2N}, \mathbb{I}\}$, then there exist elements $X'_i = X_i \mathbb{I} + \alpha^{j_i k_i} Y_{j_i} Y_{k_i}$ in the enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$ such that $\{X'_1, \ldots, X'_n\}$ span a simple algebra $\mathfrak{s}' \simeq \mathfrak{s}$ and such that $[X'_i, Y_j] = 0$ for all i, j. Therefore the insertion of these new generators X_i' into the formulae for the Casimir operators of $\mathfrak{s}$ gives the invariants sought. Moreover, the number of independent Casimir operators of $\mathfrak{g}$ is given by $\mathcal{N}(\mathfrak{g}) = \text{rank}(\mathfrak{s}) + 1$ [12].

Starting from this method, and combining it with proposition 1, we obtain a more direct matrix method for the computation of the invariants of $w\mathfrak{sp}(2N, \mathbb{R})$ from a certain polynomial associated with a $(2N + 1) \times (2N + 1)$-matrix obtained from the standard representation. Before proving the general case, we illustrate the procedure with the Lie algebra $w\mathfrak{sp}(2, \mathbb{R}) = \mathfrak{sp}(4, \mathbb{R}) \overrightarrow{\oplus}_{\Gamma_{w_1} \oplus \Gamma_0} \mathfrak{h}_2$ over the basis $\{X_{i,j}, P_k, Q_k, \mathbb{I}\}$. This algebra clearly has three invariants, one of them corresponding to the generator $\mathbb{I}$ of the centre. Using the insertion method, the operators $X'_{i,j} = X_{i,j}\mathbb{I} - P_i Q_j$, $X'_{-i,j} = X_{-i,j}\mathbb{I} - P_i P_j$, $X'_{i,-j} = X_{i,-j}\mathbb{I} - Q_i Q_j$ $(i, j = 1, 2)$ generate[1] a copy of $\mathfrak{sp}(4, \mathbb{R})$ in the enveloping algebra of $w\mathfrak{sp}(2, \mathbb{R})$. It is straightforward to verify that these operators commute with both P_k and Q_k. As a consequence, the replacement of the variables[2] $x_{i,j}$ in (19) by the new variables $x'_{i,j}$ will provide us with Casimir invariants of $w\mathfrak{sp}(2, \mathbb{R})$. To obtain these invariants we have to compute the characteristic polynomial of the matrix

$$M_2 = \begin{pmatrix} zx_{1,1} - p_1q_1 & zx_{1,2} - p_1q_2 & -zx_{-1,1} + p_1^2 & -zx_{-1,2} + p_1p_2 \\ zx_{2,1} - p_2q_1 & zx_{2,2} - p_2q_2 & -zx_{-1,2} + p_1p_2 & -zx_{-2,2} + p_2^2 \\ zx_{1,-1} - q_1^2 & zx_{1,-2} - q_1q_2 & -zx_{1,1} + p_1q_1 & -zx_{2,1} + p_2q_1 \\ zx_{1,-2} - q_1q_2 & zx_{2,-2} - q_2^2 & -zx_{1,2} + p_1q_2 & -zx_{2,2} + p_2q_2 \end{pmatrix}. \tag{21}$$

The determinant $|M_2 - T\,\mathrm{I\,d}_4|$ can be decomposed into a sum of 16 determinants, 11 of which are zero because the second summand in each column of the matrix (21) is a multiple of the column vector $(p_1, p_2, q_1, q_2)^t$. With this reduction, we obtain:

$$\begin{aligned} |M_2 - T\,\mathrm{I\,d}_4| = &\begin{vmatrix} -p_1q_1 & zx_{1,2} & -zx_{-1,1} & -zx_{-1,2} \\ -p_2q_1 & zx_{2,2} - T & -zx_{-1,2} & -zx_{-2,2} \\ -q_1^2 & zx_{1,-2} & -zx_{1,1} - T & -zx_{2,1} \\ -q_1q_2 & zx_{2,-2} & -zx_{1,2} & -zx_{2,2} - T \end{vmatrix} \\ &+ \begin{vmatrix} zx_{1,1} - T & zx_{1,2} & -zx_{-1,1} & p_1p_2 \\ zx_{2,1} & zx_{2,2} - T & -zx_{-1,2} & p_2^2 \\ zx_{1,-1} & zx_{1,-2} & -zx_{1,1} - T & p_2q_1 \\ zx_{1,-2} & zx_{2,-2} & -zx_{1,2} & p_2q_2 \end{vmatrix} \\ &+ \begin{vmatrix} zx_{1,1} - T & zx_{1,2} & p_1^2 & -zx_{-1,2} \\ zx_{2,1} & zx_{2,2} - T & p_1p_2 & -zx_{-2,2} \\ zx_{1,-1} & zx_{1,-2} & p_1q_1 & -zx_{2,1} \\ zx_{1,-2} & zx_{2,-2} & p_1q_2 & -zx_{2,2} - T \end{vmatrix} \\ &+ \begin{vmatrix} zx_{1,1} - T & -p_1q_2 & -zx_{-1,1} & -zx_{-1,2} \\ zx_{2,1} & -p_2q_2 & -zx_{-1,2} & -zx_{-2,2} \\ zx_{1,-1} & -q_1q_2 & -zx_{1,1} - T & -zx_{2,1} \\ zx_{1,-2} & -q_2^2 & -zx_{1,2} & -zx_{2,2} - T \end{vmatrix} \\ &+ \begin{vmatrix} zx_{1,1} - T & zx_{1,2} & -zx_{-1,1} & -zx_{-1,2} \\ zx_{2,1} & zx_{2,2} - T & -zx_{-1,2} & -zx_{-2,2} \\ zx_{1,-1} & zx_{1,-2} & -zx_{1,1} - T & -zx_{2,1} \\ zx_{1,-2} & zx_{2,-2} & -zx_{1,2} & -zx_{2,2} - T \end{vmatrix}. \end{aligned} \tag{22}$$

[1] Depending on the basis chosen, the expression for the new generators also changes. The basis used here differs slightly from that employed in [12].

[2] In order to obtain homogeneous polynomials in the generators, we assign the variable z to the generator $\mathbb{I}$ of the centre.

A short calculation shows that this sum can be composed as a unique determinant, namely

$$\frac{1}{T}\begin{vmatrix} zx_{1,1}-T & zx_{1,2} & -zx_{-1,1} & -zx_{-1,2} & p_1T \\ zx_{2,1} & zx_{2,2}-T & -zx_{-1,2} & -zx_{-2,2} & p_2T \\ zx_{1,-1} & zx_{1,-2} & -zx_{1,1}-T & -zx_{2,1} & q_1T \\ zx_{1,-2} & zx_{2,-2} & -zx_{1,2} & -zx_{2,2}-T & q_2T \\ -q_1 & -q_2 & p_1 & p_2 & -T \end{vmatrix}. \tag{23}$$

Determinant (23) has the advantage of avoiding all zero summands of the decomposition of (22), and is therefore more practical for computation purposes.

Repeating the same argument for the general case $N \geqslant 2$ leads to a determinant which decomposes into a sum of 2^{2N} determinants of the same order, from which $2^{2N} - 2N - 1$ are identically zero. This shows that for large values of N the method of the copy of $\mathfrak{sp}(2N, \mathbb{R})$ in the enveloping algebra of the semidirect product is not the most economical to compute the Casimir operators. The objective of this section is to prove that the reduction leading to the unique determinant (23) can be extended to the general case.

Proposition 2. *Let $N \geqslant 2$. Then the noncentral Casimir operators C_{2k+1} of $w\mathfrak{sp}(2N, \mathbb{R})$ are given by the coefficients of the polynomial*

$$|B - T\,\mathrm{Id}_{2N+1}| = T^{2N+1} + \sum_{k=1}^{N} z^{2k-1} C_{2k+1} T^{2N+1-2k}, \tag{24}$$

where B is the $(2N+1) \times (2N+1)$-matrix given by:

$$B = \begin{pmatrix} zx_{1,1} & \cdots & zx_{1,N} & -zx_{-1,1} & \cdots & -zx_{-1,N} & p_1T \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ zx_{N,1} & \cdots & zx_{N,N} & -zx_{-1,N} & \cdots & -zx_{-N,N} & p_NT \\ zx_{1,-1} & \cdots & zx_{1,-N} & -zx_{1,1} & \cdots & -zx_{N,1} & q_1T \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ zx_{1,-N} & \cdots & zx_{N,-N} & -zx_{1,N} & \cdots & -zx_{N,N} & q_NT \\ -q_1 & \cdots & -q_N & p_1 & \cdots & p_N & 0 \end{pmatrix}. \tag{25}$$

Moreover $\deg C_{2k+1} = 2k+1$ *for $k = 0, \ldots, N$.*

Proof. Since the Casimir operators of $w\mathfrak{sp}(2N, \mathbb{R})$ are obtained by replacing the generators of $\mathfrak{sp}(2N, \mathbb{R})$ by new generators spanning a copy of the symplectic algebra in the enveloping algebra into the matrix (19), the invariants are given by the following determinant:

$$\Delta = \begin{vmatrix} zx_{1,1}-p_1q_1-T & \cdots & zx_{1,N}-p_1q_N & -zx_{-1,1}+p_1^2 & \cdots & -zx_{-1,N}+p_1p_N \\ \vdots & & \vdots & \vdots & & \vdots \\ zx_{N,1}-p_Nq_1 & \cdots & zx_{N,N}-p_Nq_N-T & -zx_{-1,N}+p_1q_1 & \cdots & -zx_{-N,N}+p_N^2 \\ zx_{1,-1}-q_1^2 & \cdots & zx_{1,-N}-q_1q_N & -zx_{1,1}+p_1q_1-T & \cdots & -zx_{N,1}+p_Nq_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ zx_{1,-N}-q_1q_N & \cdots & zx_{N,-N}-q_N^2 & -zx_{1,N}+p_1q_N & \cdots & -zx_{N,N}+p_Nq_N-T \end{vmatrix}. \tag{26}$$

Now this can be simplified using the elementary rules for determinants, and taking into account that the second summand in each column of (26) is a multiple of $(p_1, \ldots, p_N, q_1, \ldots, q_N)^t$,

Δ reduces to:

$$\Delta = \begin{vmatrix} zx_{1,1}-T & \cdots & zx_{1,N} & -zx_{-1,1} & \cdots & -zx_{-1,N} \\ \vdots & & \vdots & \vdots & & \vdots \\ zx_{N,1} & \cdots & zx_{N,N}-T & -zx_{-1,N} & \cdots & -zx_{-N,N} \\ zx_{1,-1} & \cdots & zx_{1,-N} & -zx_{1,1}-T & \cdots & -zx_{N,1} \\ \vdots & & \vdots & \vdots & & \vdots \\ zx_{1,-N} & \cdots & zx_{N,-N} & -zx_{1,N} & \cdots & -zx_{N,N}-T \end{vmatrix} + \sum_{j=1}^{N}$$

$$\times \begin{vmatrix} zx_{1,1}-T & \cdots & zx_{1,j-1} & -p_1q_j & zx_{1,j+1} & \cdots & zx_{1,N} & -zx_{-1,1} & \cdots & -zx_{-1,N} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ zx_{N,1} & \cdots & zx_{N,j-1} & -p_Nq_j & zx_{N,j+1} & \cdots & zx_{N,N}-T & -zx_{-1,N} & \cdots & -zx_{-N,N} \\ zx_{1,-1} & \cdots & zx_{1,-j+1} & -q_1q_j & zx_{1,-j-1} & \cdots & zx_{1,-N} & -zx_{1,1}-T & \cdots & -zx_{N,1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ zx_{1,-N} & \cdots & zx_{j-1,-N} & -q_Nq_j & zx_{j+1,-N} & \cdots & zx_{N,-N} & -zx_{1,N} & \cdots & -zx_{N,N}-T \end{vmatrix}$$

$$+ \sum_{j=1}^{N}$$

$$\times \begin{vmatrix} zx_{1,1}-T & \cdots & zx_{1,N} & -zx_{-1,1} & \cdots & -zx_{-1,j-1} & p_1p_j & -zx_{-1,j+1} & \cdots & -zx_{-1,N} \\ \vdots & & \vdots & \vdots & & & \vdots & & & \vdots \\ zx_{N,1} & \cdots & zx_{N,N}-T & -zx_{-1,N} & \cdots & -zx_{-N,j-1} & p_Np_j & -zx_{-N,j+1} & \cdots & -zx_{-N,N} \\ zx_{1,-1} & \cdots & zx_{1,-N} & -zx_{1,1}-T & \cdots & -zx_{j-1,1} & q_1p_j & -zx_{j+1,1} & \cdots & -zx_{N,1} \\ \vdots & & \vdots & \vdots & & & & & & \vdots \\ zx_{1,-N} & \cdots & zx_{N,-N} & -zx_{1,N} & \cdots & -zx_{j-1,N} & q_Np_j & -zx_{j+1,N} & \cdots & -zx_{N,N}-T \end{vmatrix}. \tag{27}$$

The structure of these determinants suggests that they can be obtained as minors of some another determinant. Let us now consider the matrix B. The polynomial $\Delta' = |B - T\mathrm{Id}_{2N+1}|$ is given by the determinant

$$\Delta' = \begin{vmatrix} zx_{1,1}-T & \cdots & zx_{1,N} & -zx_{-1,1} & \cdots & -zx_{-1,N} & p_1T \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ zx_{N,1} & \cdots & zx_{N,N}-T & -zx_{-1,N} & \cdots & -zx_{-N,N} & p_NT \\ zx_{1,-1} & \cdots & zx_{1,-N} & -zx_{1,1}-T & \cdots & -zx_{N,1} & q_1T \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ zx_{1,-N} & \cdots & zx_{N,-N} & -zx_{1,N} & \cdots & -zx_{N,N}-T & q_NT \\ -q_1 & \cdots & -q_N & p_1 & \cdots & p_N & -T \end{vmatrix}. \tag{28}$$

Expanding it along the last row, we decompose the determinant into:

$$\Delta' = -T\left(|(B - T\mathrm{Id}_{2N+1})_{2N+1,2N+1}| + \sum_{j=1}^{N}(-1)^{2N+1+j}q_j|(B - T\mathrm{Id}_{2N+1})_{2N+1,j}|\right)$$
$$+ \left(\sum_{j=1}^{N}(-1)^{3N+1+j}p_j|(B - T\mathrm{Id}_{2N+1})_{2N+1,N+j}|\right)T, \tag{29}$$

where $(B - T\,\mathrm{I\,d}_{2N+1})_{i,j}$ is the minor of $B - T\,\mathrm{I\,d}_{2N+1}$ obtained by deleting the ith row and jth column. Inserting the variable q_j (respectively p_j) in the minor $(B - T\,\mathrm{I\,d}_{2N+1})_{2N+1,j}$ (respectively $(B - T\,\mathrm{I\,d}_{2N+1})_{2N+1,N+j}$), we recover the summands of (27). Comparison of the determinants of (26) and (28) shows that they are related as follows:

$$\Delta T + \Delta' = 0. \tag{30}$$

□

It is important to realize that the matrix B used depends on the variable T taken to evaluate the determinant $|B - T\,\mathrm{I\,d}_{2N+1}|$. Therefore we cannot speak formally of the characteristic polynomial of B, but of a polynomial closely related to it.

As illustrating examples, we consider the Lie algebras $w\mathfrak{sp}(1, \mathbb{R})$ and $w\mathfrak{sp}(2, \mathbb{R})$. The first algebra has dimension 6 and two invariants, one of them being z, the variable corresponding to the central generator $\mathbb{I}$. The noncentral invariant is determined by evaluation of determinant (28) for $N = 1$. We obtain the polynomial $T^3 + zC_3T$, where

$$C_3 = -zx_{-1,1}x_{1,-1} + x_{-1,1}q_1^2 + x_{1,-1}p_1^2 - 2x_{1,1}p_1q_1 + zx_{1,1}^2. \tag{31}$$

The symmetrization of C_3 gives the corresponding Casimir operator

$$\begin{aligned}\mathcal{S}(C_3) = {}&\left(X_{1,1}^2 - \frac{X_{-1,1}X_{1,-1} + X_{1,-1}X_{-1,1}}{2}\right)\mathbb{I} + \frac{\left(X_{-1,1}Q_1^2 + Q_1X_{-1,1}Q_1 + Q_1^2X_{-1,1}\right)}{3}\\ &- \frac{(X_{1,1}P_1Q_1 + Q_1P_1X_{1,1} + X_{1,1}Q_1P_1 + P_1Q_1X_{1,1} + Q_1X_{1,1}P_1 + P_1X_{1,1}Q_1)}{3}\\ &+ \frac{\left(P_1^2X_{1,-1} + X_{1,-1}P_1^2 + P_1X_{1,-1}P_1\right)}{3}.\end{aligned} \tag{32}$$

Observe, in particular, that the summand $\left(X_{1,1}^2 - \frac{X_{1,-1}X_{-1,1}+X_{-1,1}X_{1,-1}}{2}\right)$ corresponds to the Casimir operator of the Levi part $\mathfrak{sp}(2, \mathbb{R})$.

For the 15-dimensional Lie algebra $w\mathfrak{sp}(2, \mathbb{R})$ we find three invariants (one central). Applying (23) we get the polynomial $T^5 + zC_3T^3 + z^3C_5T$, where C_3 and C_5 are homogeneous polynomials of degrees 3 and 5, respectively. The explicit expression of the third degree invariants is

$$\begin{aligned}C_3 = {}&z\left(x_{1,1}^2 + x_{2,2}^2\right) - 2(p_1q_1x_{1,1} + p_2q_2x_{2,2} - zx_{1,2}x_{2,1} + zx_{-1,2}x_{1,-2}) + p_1^2x_{1,-1} + p_2^2x_{2,-2}\\ &+ q_1^2x_{-1,1} + q_2^2x_{-2,2} - z(x_{-2,2}x_{2,-2} + x_{-1,1}x_{1,-1})\\ &+ 2(q_1q_2x_{-1,2} + p_1p_2x_{1,-2} - p_1q_2x_{2,1} - p_2q_1x_{1,2}).\end{aligned} \tag{33}$$

The fifth degree C_5 contains 73 terms, for which reason we omit it here. The Casimir operators are again obtained using the symmetrization map $\mathcal{S}$ of (5).

3. The inhomogeneous algebras $I\mathfrak{sp}(2N, \mathbb{R})$

The previous formula (28) constitutes a simplification of determinant (26) used to determine the Casimir operators of $w\mathfrak{sp}(N, \mathbb{R})$, but its real interest lies in its application to the computation of the invariants of the inhomogeneous Lie algebras $I\mathfrak{sp}(2N, \mathbb{R})$, by virtue of the simple Inönü–Wigner contraction[3]

$$w\mathfrak{sp}(N, \mathbb{R}) \rightsquigarrow I\mathfrak{sp}(2N, \mathbb{R}) \oplus \mathbb{R} \tag{34}$$

determined by the change of basis

$$X'_{i,j} = X_{i,j}, \qquad i, j = -N, \ldots, -1, 1, \ldots, N; \quad \mathbb{I}' = \mathbb{I} \tag{35}$$

[3] For the definition of simple IW-contraction used here see e.g. [21].

$$P_i' = \frac{1}{\sqrt{t}} P_i, \quad i = 1, \ldots, N; \qquad Q_i' = \frac{1}{\sqrt{t}} Q_i, \quad i = 1, \ldots, N \tag{36}$$

for $t \to \infty$, but this process rapidly becomes tedious with large N. It is therefore convenient to develop a direct method to obtain the Casimir operators, independently of the limiting process of (35) and (36). These algebras have, as is known, N independent Casimir operators [22].

Proposition 3. *Let $N \geqslant 2$. Then the Casimir operators C_{2k} of $I\mathfrak{sp}(2N,\mathbb{R})$ are given by the coefficients of the polynomial*

$$|C - T\,\mathrm{Id}_{2N+1}| + |A - T\,\mathrm{Id}_{2N}|T = \sum_{k=1}^{N} C_{2k+1} T^{2N+1-2k}, \tag{37}$$

where

$$C = \begin{pmatrix} x_{1,1} & \cdots & x_{1,N} & -x_{-1,1} & \cdots & -x_{-1,N} & p_1 T \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ x_{N,1} & \cdots & x_{N,N} & -x_{-1,N} & \cdots & -x_{-N,N} & p_N T \\ x_{1,-1} & \cdots & x_{1,-N} & -x_{1,1} & \cdots & -x_{N,1} & q_1 T \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ x_{1,-N} & \cdots & x_{N,-N} & -x_{1,N} & \cdots & -x_{N,N} & q_N T \\ -q_1 & \cdots & -q_N & p_1 & \cdots & p_N & 0 \end{pmatrix}. \tag{38}$$

Moreover $\deg C_{2k+1} = 2k+1$.

Proof. The proof is essentially the same as that of proposition 2. By the contraction, the invariants of the inhomogeneous algebras are obtained from the limit for $t \to \infty$ of the invariants of $w\mathfrak{sp}(N,\mathbb{R})$, and are given by:

$$\lim_{t\to\infty} \frac{1}{t} \times \begin{vmatrix} zx_{1,1} - tp_1q_1 - T & \cdots & zx_{1,N} - tp_1q_N & -zx_{-1,1} + tp_1^2 & \cdots & -zx_{-1,N} + tp_1p_N \\ \vdots & & \vdots & \vdots & & \vdots \\ zx_{N,1} - tp_Nq_1 & \cdots & zx_{N,N} - tp_Nq_N - T & -zx_{-1,N} + tp_1q_1 & \cdots & -zx_{-N,N} + tp_N^2 \\ zx_{1,-1} - tq_1^2 & \cdots & zx_{1,-N} - tq_1q_N & -zx_{1,1} + tp_1q_1 - T & \cdots & -zx_{N,1} + tp_Nq_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ zx_{1,-N} - tq_1q_N & \cdots & zx_{N,-N} - tq_N^2 & -zx_{1,N} + tp_1q_N & \cdots & -zx_{N,N} + tp_Nq_N - T \end{vmatrix}. \tag{39}$$

Reducing the determinant by standard methods and taking the limit, we obtain the following sum:

$$\sum_{j=1}^{N} \times \begin{vmatrix} zx_{1,1} - T & \cdots & zx_{1,j-1} & -p_1q_j & zx_{1,j+1} & \cdots & zx_{1,N} & -zx_{-1,1} & \cdots & -zx_{-1,N} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ zx_{N,1} & \cdots & zx_{N,j-1} & -p_Nq_j & zx_{N,j+1} & \cdots & zx_{N,N} - T & -zx_{-1,N} & \cdots & -zx_{-N,N} \\ zx_{1,-1} & \cdots & zx_{1,-j+1} & -q_1q_j & zx_{1,-j-1} & \cdots & zx_{1,-N} & -zx_{1,1} - T & \cdots & -zx_{N,1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ zx_{1,-N} & \cdots & zx_{j-1,-N} & -q_Nq_j & zx_{j+1,-N} & \cdots & zx_{N,-N} & -zx_{1,N} & \cdots & -zx_{N,N} - T \end{vmatrix}$$

$$
+\sum_{j=1}^{N}
\times\begin{vmatrix}
zx_{1,1}-T & \cdots & zx_{1,N} & -zx_{-1,1} & \cdots & -zx_{-1,j-1} & p_1p_j & -zx_{-1,j+1} & \cdots & -zx_{-1,N}\\
\vdots & & \vdots & \vdots & & & \vdots & & & \vdots\\
zx_{N,1} & \cdots & zx_{N,N}-T & -zx_{-1,N} & \cdots & -zx_{-N,j-1} & p_Np_j & -zx_{-N,j+1} & \cdots & -zx_{-N,N}\\
zx_{1,-1} & \cdots & zx_{1,-N} & -zx_{1,1}-T & \cdots & -zx_{j-1,1} & p_1q_j & -zx_{j+1,1} & \cdots & -zx_{N,1}\\
\vdots & & \vdots & \vdots & & & & & & \vdots\\
zx_{1,-N} & \cdots & zx_{N,-N} & -zx_{1,N} & \cdots & -zx_{j-1,N} & p_Nq_j & -zx_{j+1,N} & \cdots & -zx_{N,N}-T
\end{vmatrix}. \tag{40}
$$

This sum is very similar to that of (27), up to the fact that here all involved determinants have a column whose entries are products of the variables p_i and q_j associated with the standard representation Γ_{ω_1} of $\mathfrak{sp}(2N,\mathbb{R})$. Expanding the sum, we obtain a polynomial:

$$
\sum_{k=1}^{N} z^{2k-1}C_{2k+1}T^{2N-2k}, \tag{41}
$$

where the C_{2k+1} are homogeneous polynomials of degree $2k+1$. The invariants of $I\mathfrak{sp}(2N,\mathbb{R})$ are given by the C_{2k+1}, while z is the invariant of the direct summand $\mathbb{R}$ of the contraction.

If we now expand the determinant $|C-T\mathrm{I\,d}_{2n+1}|$ (compare with (27)), we get the decomposition

$$
\begin{aligned}
|C-T\mathrm{I\,d}_{2N+1}| &= -T\left(|(C-T\mathrm{I\,d}_{2N+1})_{2N+1,2N+1}| + \sum_{j=1}^{N}(-1)^{2N+1+j}q_j|(C-T\mathrm{I\,d}_{2N+1})_{2N+1,j}|\right)\\
&\quad + \left(\sum_{j=1}^{N}(-1)^{3N+1+j}p_j|(C-T\mathrm{I\,d}_{2N+1})_{2N+1,N+j}|\right)T.
\end{aligned} \tag{42}
$$

Proceeding as before, it is not difficult to see that the sum

$$
\sum_{j=1}^{N}((-1)^{2N+1+j}q_j|(C-T\mathrm{I\,d}_{2N+1})_{2N+1,j}| + (-1)^{3N+1+j}p_j|(C-T\mathrm{I\,d}_{2N+1})_{2N+1,N+j}|))T \tag{43}
$$

coincides with the sum (40) when we set $z=1$. The remaining summand $|(C-T\mathrm{I\,d}_{2N+1})_{2N+1,2N+1}|$ is nothing but the characteristic polynomial of the matrix A of $\mathfrak{sp}(2N,\mathbb{R})$ associated with the standard representation (see (19)) multiplied by T. Therefore the sum

$$
|C-T\mathrm{I\,d}_{2N+1}| + |A-T\mathrm{I\,d}_{2N}|T = \sum_{k=1}^{N} C_{2k+1}T^{2N+1-2k},
$$

gives the Casimir operators of the inhomogeneous algebra. □

The advantage of this determinantal procedure for the invariants of $\mathfrak{sp}(2N,\mathbb{R})$ in comparison with the contraction or embedding methods usually applied in the literature is remarkable, since we are only using the structure tensor of the algebra, and not specific realizations of it.

As examples, we evaluate the invariants of the inhomogeneous algebras $I\mathfrak{sp}(N,\mathbb{R})$ for $N=1,2$. In the first case, the only invariant is given by

$$
|C-T\mathrm{I\,d}_3| + |A-T\mathrm{I\,d}_2| = \begin{vmatrix} x_{1,1}-T & -x_{-1,1} & p_1T\\ x_{1,-1} & -x_{1,1}-T & q_1T\\ -q_1 & p_1 & -T\end{vmatrix} + T\begin{vmatrix} x_{1,1}-T & -x_{-1,1}\\ x_{1,-1} & -x_{1,1}-T\end{vmatrix} = C_3T,
$$

where

$$C_3 = x_{1,2}q_1^2 + x_{2,1}p_1^2 - 2x_{1,1}p_1q_1. \tag{44}$$

We now apply the symmetrization map $\mathcal{S}$ to recover the Casimir operator:

$$\mathcal{S}(C_3) = \frac{\left(X_{1,2}Q_1^2 + Q_1X_{1,2}Q_1 + Q_1^2X_{1,2}\right)}{3} + \frac{\left(X_{2,1}P_1^2 + P_1X_{2,1}P_1 + P_1^2X_{2,1}\right)}{3} - \frac{(X_{1,1}P_1Q_1 + Q_1P_1X_{1,1} + X_{1,1}Q_1P_1 + P_1Q_1X_{1,1} + Q_1X_{1,1}P_1 + P_1X_{1,1}Q_1)}{3}. \tag{45}$$

Observe that this operator is the difference of (32) and the operator of $\mathfrak{sp}(2,\mathbb{R})$. For the inhomogeneous algebra $I\mathfrak{sp}(4,\mathbb{R})$ formula (37) gives two invariants:

$$\begin{aligned} C_3 = {} & -2(p_1q_1x_{1,1} + p_2q_2x_{2,2} - zx_{1,2}x_{2,1} + zx_{-1,2}x_{1,-2}) + p_1^2x_{1,-1} + p_2^2x_{2,-2} \\ & + q_1^2x_{-1,1} + q_2^2x_{-2,2} + 2(q_1q_2x_{-1,2} + p_1p_2x_{1,-2} - p_1q_2x_{2,1} - p_2q_1x_{1,2}), \end{aligned} \tag{46}$$

and

$$\begin{aligned} C_5 = {} & \left(x_{2.2}(x_{1,-1}x_{2,2} - 2x_{2,1}x_{1,-2}) + x_{-2,2}\left(x_{1,-2}^2 - x_{1,-1}x_{2,-2}\right) + x_{2,1}^2x_{2,-2}\right)p_1^2 \\ & + \left(-2x_{1,1}x_{1,-2}x_{1,2} + x_{1,-1}x_{1,2}^2 + x_{1,-2}^2x_{-1,1} - x_{1,-1}x_{2,-2}x_{-1,1} + x_{1,1}^2x_{2,-2}\right)p_2^2 \\ & + \left(x_{2,2}^2x_{-1,1} - 2x_{1,2}x_{-1,2}x_{2,2} + x_{2,-2}x_{-1,2}^2 - x_{2,-2}x_{-1,1}x_{-2,2} + x_{1,2}^2x_{-2,2}\right)q_1^2 \\ & + \left(x_{1,-1}x_{-1,2}^2 + x_{2,1}^2x_{-1,1} - x_{1,-1}x_{-1,1}x_{-2,2} - 2x_{1,1}x_{-1,2}x_{2,1} + x_{1,1}^2x_{-2,2}\right)q_2^2 \\ & + 2(x_{2,1}(x_{1,1}x_{2,2} + x_{1,-2}x_{-1,2} - x_{2,1}x_{1,2}) + x_{-2,2}(x_{1,-1}x_{1,2} - x_{1,1}x_{1,-2}) \\ & - x_{1,-1}x_{2,2}x_{-1,2})p_1q_2 + 2(x_{2,2}(x_{2,1}x_{1,2} - x_{1,1}x_{2,2} + x_{1,-2}x_{-1,2}) \\ & + x_{-2,2}(x_{1,1}x_{2,-2} - x_{1,-2}x_{1,2}) - x_{2,1}x_{2,-2}x_{-1,2})p_1q_1 \\ & + 2(x_{1,-2}(x_{2,1}x_{1,2} - x_{1,-2}x_{-1,2} + x_{1,1}x_{2,2}) + x_{2,-2}(x_{1,-1}x_{-1,2} - x_{1,1}x_{2,1}) \\ & - x_{1,-1}x_{1,2}x_{2,2})p_1p_2 + 2(x_{-1,2}(x_{1,1}x_{2,2} - x_{1,-2}x_{-1,2} + x_{2,1}x_{1,2}) \\ & + x_{-1,1}(x_{1,-2}x_{-2,2} - x_{2,1}x_{2,2}) - x_{1,1}x_{1,2}x_{-2,2})q_1q_2 \\ & + 2(x_{1,1}(x_{1,2}x_{2,1} + x_{1,-2}x_{-1,2} - x_{1,1}x_{2,2}) + x_{-1,1}(x_{1,-1}x_{2,2} - x_{2,1}x_{1,-2}) \\ & - x_{1,-1}x_{1,2}x_{-1,2})p_2q_2 + 2(x_{1,2}(x_{1,-2}x_{-1,2} + x_{1,1}x_{2,2} - x_{2,1}x_{1,2}) \\ & + (x_{2,1}x_{2,-2} - x_{1,-2}x_{2,2})x_{-1,1} - x_{1,1}x_{2,-2}x_{-1,2})p_2q_1. \end{aligned} \tag{47}$$

Symmetrizing these functions we obtain two independent Casimir operators of $I\mathfrak{sp}(4,\mathbb{R})$.

4. Relation of C to a faithful representation of $I\mathfrak{sp}(2N,\mathbb{R})$

Formula (37) for the invariants of the Lie algebras $I\mathfrak{sp}(2N,\mathbb{R})$ is of special interest, not only because of its simplicity, but also because, in a certain sense, the matrix C of (38) used for its computation is related to a faithful representation of the inhomogeneous algebras.

Proposition 4. *For $N \geqslant 2$ the matrix C of (38) decomposes as*

$$C = M_N\begin{pmatrix} \mathrm{Id}_{2N} & 0 \\ 0 & T \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ -q_1 & \cdots & -q_N & p_1 & \cdots & p_N & 0 \end{pmatrix}, \tag{48}$$

where

$$M_N := \begin{pmatrix} x_{1,1} & \cdots & x_{1,N} & -x_{-1,1} & \cdots & -x_{-1,N} & p_1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ x_{N,1} & \cdots & x_{N,N} & -x_{-1,N} & \cdots & -x_{-N,N} & p_N \\ x_{1,-1} & \cdots & x_{1,-N} & -x_{1,1} & \cdots & -x_{N,1} & q_1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ x_{1,-N} & \cdots & x_{N,-N} & -x_{1,N} & \cdots & -x_{N,N} & q_N \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{49}$$

Moreover, M_N defines a $(2N+1)$-dimensional faithful representation of $I\mathfrak{sp}(2N,\mathbb{R})$.

Proof. The proof of the decomposition follows at once. Let $I_{i,j}$ be the elementary matrix whose entry is 1 in the ith row and jth column, and zero elsewhere. For generators (9)–(11) define the mapping[4]

$$\begin{aligned} &\Phi(X_{i,j}) = I_{ij} - I_{j+N,i+N}; && 1 \leqslant i \leqslant j \leqslant N \\ &\Phi(X_{-i,i}) = -2I_{i,N+i}; \quad \Phi(X_{i,-i}) = 2I_{N+i,i}; && 1 \leqslant i \leqslant N \\ &\Phi(X_{-i,j}) = -(I_{i,N+j} + I_{j,N+i}); && 1 \leqslant i < j \leqslant N \\ &\Phi(X_{i,-j}) = (I_{N+i,j} + I_{N+j,i}); && 1 \leqslant i < j \leqslant N \\ &\Phi(P_i') = I_{i,2N+1}; \quad \Phi(Q_i') = I_{N+i,2N+1}; && 1 < i \leqslant N. \end{aligned} \tag{50}$$

It is straightforward to verify that the matrix commutator satisfies relations (12)–(17), thus define a representation of $I\mathfrak{sp}(2N,\mathbb{R})$. Since no element is mapped onto the zero matrix, Φ is faithful. □

We thus recover the usual standard representation of $I\mathfrak{sp}(2N,\mathbb{R})$. Equation (37) can be seen as the adaptation of the classical Gel'fand method to the computation of invariants of inhomogeneous algebras.

5. The extended Schrödinger algebra

In this section we focus on an extremely important Lie algebra that arises as a subalgebra of the semidirect product $w\mathfrak{sp}(N,\mathbb{R})$, the Schrödinger algebra $\widehat{S}(N)$. First introduced in [4, 23], the invariance algebra of the Schrödinger equation in $(N+1)$-dimensional space time has attracted considerable interest in recent physical literature ([24, 25] and references therein). The Schrödinger algebra $\widehat{S}(N)$ in $(N+1)$-dimensional spacetime is a $\frac{1}{2}(N^2+3N+8)$-dimensional Lie algebra with non-trivial commutators

$$\begin{aligned} &[J_{\mu\nu}, J_{\lambda\sigma}] = \delta_{\mu\lambda}J_{\nu\sigma} + \delta_{\nu\sigma}J_{\mu\lambda} - \delta_{\mu\sigma}J_{\nu\lambda} - \delta_{\nu\lambda}J_{\mu\sigma}, \\ &[J_{\mu\nu}, R_\lambda] = \delta_{\mu\lambda}R_\nu - \delta_{\nu\lambda}R_\mu, && [J_{\mu\nu}, G_\lambda] = \delta_{\mu\lambda}G_\nu - \delta_{\nu\lambda}G_\mu, \\ &[P_t, G_\mu] = P_\mu, && [K, R_\mu] = -G_\mu, \\ &[D, G_\mu] = G_\mu, && [D, R_\mu] = -R_\mu, \\ &[D, K] = 2K, && [D, P_t] = -2P_t, \\ &[K, P_t] = -D. && [R_\mu, G_\nu] = \delta_{\mu\nu}M \end{aligned} \tag{51}$$

[4] According to contraction (35), (36), the brackets for the generators P_i' and Q_i' of $I\mathfrak{sp}(2N,\mathbb{R})$ are given by

$$[P_i', Q_i'] = \lim_{t\to\infty}[P_i', Q_i'] = 0.$$

over the basis $\{J_{ij}, R_k, G_k, K, D, P_t, M\}$, where $J_{\mu\nu} + J_{\nu\mu} = 0$ are rotations, R_μ the spatial translation generators, P_t the time translation, G_μ special Galilei transformations, D the generator of scale transformations, K the generator of Galilean conformal transformations and M commutes with all generators [4, 23]. The quotient by the centre, generated by M, gives the unextended Schrödinger algebra $S(N)$. It follows from the brackets that $\widehat{S}(N)$ is the semidirect product of a semisimple Lie algebra, isomorphic to $\mathfrak{so}(N)\oplus\mathfrak{sl}(2,\mathbb{R})$, with the Heisenberg algebra $\mathfrak{h}_N$ of dimension $(2N+1)$, i.e., its Levi decomposition is $(\mathfrak{so}(N)\oplus\mathfrak{sl}(2,\mathbb{R}))\overrightarrow{\oplus}_R\mathfrak{h}_N$, where the representation R can be identified with $\left(D_{\frac{1}{2}}\otimes\Lambda\right)\oplus D_0$. Here $D_{\frac{1}{2}}\otimes\Lambda$ is the tensor product of the standard representations $D_{\frac{1}{2}}$ of $\mathfrak{sl}(2,\mathbb{R})$ and Λ of $\mathfrak{so}(N)$, respectively [25], and D_0 is the trivial representation. Moreover, $\widehat{S}(N)$ is isomorphic to a subalgebra of $\mathfrak{wsp}(N,\mathbb{R})$ by means of the homomorphism of Lie algebras defined by

$$\Phi(J_{\mu\nu}) = X_{\mu,\nu} - X_{\nu,\mu}, \qquad \Phi(G_\mu) = P_\mu, \qquad \Phi(R_\mu) = Q_\mu, \qquad 1\leqslant\mu<\nu\leqslant N, \tag{52}$$

$$\Phi(K) = \frac{1}{2}\sum_{\mu=1}^{N} X_{-\mu,\mu}, \qquad \Phi(P_t) = \frac{1}{2}\sum_{\mu=1}^{N} X_{\mu-,\mu}, \qquad \Phi(D) = \sum_{\mu=1}^{N} X_{\mu,\mu}. \tag{53}$$

For $N\geqslant 2$ the inclusion is strict, while $\widehat{S}(1)$ and $\mathfrak{wsp}(1,\mathbb{R})$ are isomorphic.

One can ask whether the matrix method exhibited for the symplectic algebras can be enlarged and adapted to compute the Casimir operators of the Schrödinger algebra. The answer is in the affirmative, but due to the structure of the Levi part, we will have to divide the method into two steps. To see why this division is convenient, we consider the system of PDEs (4) corresponding to $\widehat{S}(N)$ over the preceding basis:

$$\widehat{J}_{\mu\nu}F = j_{\nu\sigma}\frac{\partial F}{\partial j_{\mu\sigma}} - j_{\nu\lambda}\frac{\partial F}{\partial j_{\lambda\mu}} - j_{\mu\sigma}\frac{\partial F}{\partial j_{\nu\sigma}} + j_{\mu\lambda}\frac{\partial F}{\partial j_{\lambda\nu}} + r_\nu\frac{\partial F}{\partial r_\mu} - r_\mu\frac{\partial F}{\partial r_\nu} + g_\nu\frac{\partial F}{\partial g_\mu} - g_\mu\frac{\partial F}{\partial g_\nu} = 0 \tag{54}$$

$$\widehat{D}F = 2k\frac{\partial F}{\partial k} - 2p_t\frac{\partial F}{\partial p_t} + g_\mu\frac{\partial F}{\partial g_\mu} - r_\mu\frac{\partial F}{\partial r_\mu} = 0 \tag{55}$$

$$\widehat{K}F = -2k\frac{\partial F}{\partial d} - d\frac{\partial F}{\partial p_t} - g_\mu\frac{\partial F}{\partial r_\mu} = 0 \tag{56}$$

$$\widehat{P}_tF = 2p_t\frac{\partial F}{\partial d} + d\frac{\partial F}{\partial k} + r_\mu\frac{\partial F}{\partial g_\mu} = 0 \tag{57}$$

$$\widehat{G}_\mu F = -g_\nu\frac{\partial F}{\partial j_{\mu\nu}} + g_\nu\frac{\partial F}{\partial j_{\nu\mu}} - r_\mu\frac{\partial F}{\partial p_t} - g_\mu\frac{\partial F}{\partial d} - m\frac{\partial F}{\partial r_\mu} = 0 \tag{58}$$

$$\widehat{R}_\mu F = -r_\nu\frac{\partial F}{\partial j_{\mu\nu}} + r_\nu\frac{\partial F}{\partial j_{\nu\mu}} + g_\mu\frac{\partial F}{\partial k} + r_\mu\frac{\partial F}{\partial d} + m\frac{\partial F}{\partial g_\mu} = 0, \tag{59}$$

where $1\leqslant\mu,\nu,\lambda,\sigma\leqslant N$. We will see that a maximal set of independent Casimir invariants of (54)–(59) can be obtained from the invariants of two particular subalgebras. Consider $\mathfrak{g}_1 = \mathfrak{so}(N)\overrightarrow{\oplus}_{2\Lambda}\mathfrak{h}_N$ and $\mathfrak{g}_2 = \mathfrak{sl}(2,\mathbb{R})\overrightarrow{\oplus}_{nD_{\frac{1}{2}}}\mathfrak{h}_N$, which are the semidirect products of the Heisenberg algebra with the components of the Levi part of $\widehat{S}(N)$. Observe that for $N=2$ the algebra $\mathfrak{g}_1$ is solvable since $\mathfrak{so}(2)$ is Abelian. The following result shows that the Schrödinger algebra has invariants in common with the subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

Lemma 1. *For any $N\geqslant 2$, the invariants of the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are also invariants of the Schrödinger algebra $\widehat{S}(N)$.*

Proof. For the Lie algebra $\mathfrak{g}_1$, the invariants are obtained from the system of PDEs:

$$\widehat{J}_{\mu\nu}F = j_{\nu\sigma}\frac{\partial F}{\partial j_{\mu\sigma}} - j_{\nu\lambda}\frac{\partial F}{\partial j_{\lambda\mu}} - j_{\mu\sigma}\frac{\partial F}{\partial j_{\nu\sigma}} + j_{\mu\lambda}\frac{\partial F}{\partial j_{\lambda\nu}} + r_\nu\frac{\partial F}{\partial r_\mu} - r_\mu\frac{\partial F}{\partial r_\nu} + g_\nu\frac{\partial F}{\partial g_\mu} - g_\mu\frac{\partial F}{\partial g_\nu} = 0 \tag{60}$$

$$\widehat{G}_\mu F = -g_\nu\frac{\partial F}{\partial j_{\mu\nu}} + g_\nu\frac{\partial F}{\partial j_{\nu\mu}} - m\frac{\partial F}{\partial r_\mu} = 0 \tag{61}$$

$$\widehat{P}_\mu F = -r_\nu\frac{\partial F}{\partial j_{\mu\nu}} + r_\nu\frac{\partial F}{\partial j_{\nu\mu}} + m\frac{\partial F}{\partial g_\mu} = 0. \tag{62}$$

Observe that the equations corresponding to the rotations are the same as (54), while those corresponding to the generators G_μ and R_μ are a summand of equations (58), respectively (59). Thus any solution of (60)–(62) is a particular solution of (54)–(59). Since the algebra $\mathfrak{g}_1$ is nothing but the semidirect product of the orthogonal algebra $\mathfrak{so}(N)$ with a Heisenberg algebra $\mathfrak{h}_N$, it follows from Quesne's theorem that for $N \geqslant 3$ the number of Casimir operators is given by $1 + \left[\frac{N}{2}\right]$. For $N = 2$ it is trivial to see that $\mathcal{N}(\mathfrak{g}_1) = 2$. The corresponding analysis for the subalgebra $\mathfrak{g}_2$ is very similar. The system associated with $\mathfrak{g}_2$ is:

$$\widehat{D}F = 2k\frac{\partial F}{\partial k} - 2p_t\frac{\partial F}{\partial p_t} + g_\mu\frac{\partial F}{\partial g_\mu} - r_\mu\frac{\partial F}{\partial r_\mu} = 0 \tag{63}$$

$$\widehat{K}F = -2k\frac{\partial F}{\partial d} - d\frac{\partial F}{\partial p_t} - g_\mu\frac{\partial F}{\partial r_\mu} = 0 \tag{64}$$

$$\widehat{P}_t F = 2p_t\frac{\partial F}{\partial d} + d\frac{\partial F}{\partial k} + r_\mu\frac{\partial F}{\partial g_\mu} = 0 \tag{65}$$

$$\widehat{G}_\mu F = -r_\mu\frac{\partial F}{\partial p_t} - g_\mu\frac{\partial F}{\partial d} - m\frac{\partial F}{\partial r_\mu} = 0 \tag{66}$$

$$\widehat{R}_\mu F = g_\mu\frac{\partial F}{\partial k} + r_\mu\frac{\partial F}{\partial d} + m\frac{\partial F}{\partial g_\mu} = 0. \tag{67}$$

Again any solution of this system provides a particular solution of (54)–(59). Since rank $(\mathfrak{sl}(2,\mathbb{R})) = 1$, the algebra $\mathfrak{g}_2$ has two invariants for any N. Observe that the solution z is a common invariant to both subalgebras. □

This lemma shows that we obtain $2 + \left[\frac{N}{2}\right]$ independent invariants for the Schrödinger algebra. It remains to prove these form a complete set of invariants, i.e., there are no additional independent solutions of system (54)–(59).

Lemma 2. *For any $N \geqslant 1$ the Schrödinger algebra $\widehat{S}(N)$ satisfies the equality $\mathcal{N}(\widehat{S}(N)) = 2 + \left[\frac{N}{2}\right]$.*

Proof. It suffices to prove the existence of a suitable non-trivial contraction having $2 + \left[\frac{N}{2}\right]$ invariants. Let Ψ be the automorphism of $\widehat{S}(N)$ defined by

$$\Psi(D) = \frac{1}{t^2}D, \qquad \Psi(K) = \frac{1}{t}K, \qquad \Psi(P_t) = \frac{1}{t}P_t, \tag{68}$$

$$\Psi(J_{\mu\nu}) = J_{\mu\nu}, \qquad \Psi(G_\mu) = G_\mu, \qquad \Psi(R_\mu) = R_\mu. \tag{69}$$

Since only the generators of $\mathfrak{sl}(2,\mathbb{R})$ are being rescaled, the subalgebra $\mathfrak{g}_1$ will be invariant by the corresponding contraction, while for $t \to \infty$ the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra contracts onto the three-dimensional Heisenberg algebra. Therefore the contraction defined by Ψ is isomorphic to the direct sum $(\mathfrak{so}(N)\overrightarrow{\oplus}_{2\Lambda}\mathfrak{h}_N,)\oplus\mathfrak{h}_1$. By the properties of contractions we have

$$2+\left[\frac{N}{2}\right]\leqslant\mathcal{N}(\widehat{S}(N))\leqslant\mathcal{N}((\mathfrak{so}(N)\overrightarrow{\oplus}_{2\Lambda}\mathfrak{h}_N,)\oplus\mathfrak{h}_1). \tag{70}$$

But the number of invariants of the contracton is given by

$$\mathcal{N}((\mathfrak{so}(N)\overrightarrow{\oplus}_R\mathfrak{h}_N)\oplus\mathfrak{h}_1)=\mathcal{N}(\mathfrak{so}(N)\overrightarrow{\oplus}_{2\Lambda}\mathfrak{h}_N)+1, \tag{71}$$

and by the preceding lemma we have $\mathcal{N}(\mathfrak{so}(N)\overrightarrow{\oplus}_R\mathfrak{h}_N)=1+\left[\frac{N}{2}\right]$. □

With these results we have reduced the problem of finding the Casimir operators of the Schrödinger algebra to that of finding the invariants of the subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. For the first case we will develop a matrix method similar to that given for the symplectic algebras, while for $\mathfrak{g}_2$ the result follows at once from the determinantal formulae developed in [6].

Proposition 5. *For any $N\geqslant 2$, the noncentral Casimir operators $C_{2k+2}\left(1\leqslant k\leqslant\left[\frac{N}{2}\right]\right)$ of the Lie algebra $\mathfrak{so}(N)\overrightarrow{\oplus}_{2\Lambda}\mathfrak{h}_N$ are given by the coefficients of the polynomial*

$$|D_N-T\,\mathrm{Id}_{N+2}|=T^{N+2}+\sum_{k=1}^{\left[\frac{N}{2}\right]}T^{N+2-2k}m^{2k-2}C_{2k+2}, \tag{72}$$

where

$$D_N=\begin{pmatrix}0 & -mj_{12} & \cdots & -mj_{1N} & g_1T & r_1T\\ mj_{12} & 0 & \cdots & -mj_{2N} & g_2T & r_2T\\ \vdots & \vdots & & \vdots & \vdots & \vdots\\ mj_{1N} & mj_{2N} & \cdots & 0 & g_NT & r_NT\\ -r_1 & -r_2 & \cdots & -r_N & 0 & 0\\ g_1 & g_2 & \cdots & g_N & 0 & 0\end{pmatrix}. \tag{73}$$

Moreover, $\deg C_{2k+2}=2k+2$ *for any k.*

Proof. The argument is still the same as in propositions 1 and 2. Gel'fand proved in [7] that the Casimir operators of the orthogonal algebras $\mathfrak{so}(N)$ are obtained from the characteristic polynomial of the matrix

$$M_N(j_{\mu\nu})=\begin{pmatrix}0 & -j_{12} & \cdots & -j_{1N}\\ j_{12} & 0 & \cdots & -j_{2N}\\ \vdots & \vdots & & \vdots\\ j_{1N} & j_{2N} & \cdots & 0\end{pmatrix}. \tag{74}$$

Now we search for operators $J'_{\mu\nu}$ in the enveloping algebra of $\mathfrak{so}(N)\overrightarrow{\oplus}_{2\Lambda}\mathfrak{h}_N$ that span an algebra isomorphic to $\mathfrak{so}(N)$ and commute with all generators G_μ and R_μ. This can be done by using the reduction method developed in [13], and we obtain:

$$J'_{\mu\nu}M+(G_\mu R_\nu-R_\mu G_\nu),\qquad 1\leqslant\mu<\nu\leqslant N. \tag{75}$$

Considering representation (3), these operators correspond to the variables

$$j'_{\mu\nu}m+(g_\mu r_\nu-r_\mu g_\nu),\qquad 1\leqslant\mu<\nu\leqslant N. \tag{76}$$

Replacing $j_{\mu\nu}$ in (74) by the new variables (76), we obtain the invariants of $\mathfrak{so}(N)\overrightarrow{\oplus}_{2\Lambda}\mathfrak{h}_N$ from the characteristic polynomial $|M_N(j'_{\mu\nu})-T\,\mathrm{Id}_N|$. Now this determinant can be simplified using elementary techniques, and after some manipulation it follows that

$$T^2|M_N(j'_{\mu\nu}) - T\,\mathrm{Id}_N| - |D_N(j'_{\mu\nu}) - T\,\mathrm{Id}_{N+2}| = 0. \tag{77}$$

□

The analogous matrix formula for the invariants of $\mathfrak{g}_2$ is merely a reformulation of the formula developed in [6], so we omit the detailed proof.

Proposition 6. *For any $N \geqslant 1$ the noncentral Casimir operators of $\mathfrak{sl}(2,\mathbb{R})\overrightarrow{\oplus}_R\mathfrak{h}_N$ are given by the coefficients of the polynomial*

$$|E_N - T\,\mathrm{Id}_{N+2}| = T^{N+2} + C'_4T^N, \tag{78}$$

where

$$E_N = \begin{pmatrix} md & -2mk & g_1T & \cdots & g_NT \\ 2mp_t & -md & r_1T & \cdots & r_NT \\ -r_1 & g_1 & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \cdots & 0 \\ -r_N & g_N & 0 & \cdots & 0 \end{pmatrix}. \tag{79}$$

Morover C'_4 is a homogeneous polynomial of degree 4.

These formulae have an important consequence: for any $N \geqslant 2$ the Schrödinger algebra $\widehat{S}(N)$ has two Casimir operators of order four, namely C_4 and C'_4. For this reason the Casimir operators will not be expressible in terms of a unique matrix, but using both matrices D_N and E_n. Taking together these results, the procedure to determine a maximal set of Casimir operators for the Schrödinger algebra $\widehat{S}(N)$ can be summarized in the following steps:

(i) Determine the $\left[\frac{N}{2}\right]$ coefficients C_{2k+2} of the polynomial $|D_N - T\,\mathrm{Id}_{N+2}| = T^{N+2} + \sum_{k=1}^{\left[\frac{N}{2}\right]} T^{N+2-2k}m^{2k-2}C_{2k+2}$,
(ii) Determine the coefficient C'_4 of the polynomial $|E_N - T\,\mathrm{Id}_{N+2}| = T^{N+2} + C'_4T^N$,
(iii) Take the central variable z,
(iv) Symmetrize the functions C_{2k+2}, C'_4 and z using the mapping $\mathcal{S}$ of (5).

As examples we compute the invariants of the Schrödinger algebra $\widehat{S}(N)$ for $N = 2, 3$. For the nine-dimensional algebra $\widehat{S}(2)$ we have $\mathcal{N}(\widehat{S}(2)) = 3$. The non-central are obtained using formulae (72) and (78). Expanding the determinants we find the polynomials

$$C_4 = (j_{12}m + (g_1p_2 - g_2p_1))^2 \tag{80}$$

$$C'_4 = (g_1p_2 - g_2p_1)^2 - m^2(d^2 - 4kp_t) - 2mp_t\left(g_1^2 + g_2^2\right) - 2mk\left(p_1^2 + p_2^2\right) - 2md(p_1g_1 + p_2g_2). \tag{81}$$

It follows in particular that C_4 is a square, so that $\widehat{S}(2)$ has an invariant of degree two.

The Lie algebra $\widehat{S}(3)$ also has three independent invariants, as follows from lemma 2. In this case, the formulae provide the following non-central invariants:

$$C_4 = (g_1p_2 - g_2p_1)^2 + (g_1p_3 - g_3p_1)^2 + (g_2p_3 - g_3p_2)^2 + m^2(j_{12}^2 + j_{13}^2 + j_{23}^2) - 2m(j_{12}(g_1p_2 - g_2p_1) + j_{13}(g_1p_3 - g_3p_1) + j_{23}(g_2p_3 - g_3p_2)). \tag{82}$$

$$C'_4 = (g_1p_2 - g_2p_1)^2 + (g_1p_3 - g_3p_1)^2 + (g_2p_3 - g_3p_2)^2 + m^2(4kp_t - d^2) - 2m\left(k\left(p_1^2 + p_2^2 + p_3^2\right) + p_t\left(g_1^2 + g_2^2 + g_3^2\right) + d(p_1g_1 + p_2g_2 + p_3g_3)\right). \tag{83}$$

6. Applications to other inhomogeneous Lie algebras

The preceding sections show how the Casimir operators of the semidirect products $w\mathfrak{sp}(N,\mathbb{R})$ and the inhomogeneous Lie algebras $I\mathfrak{sp}(2N,\mathbb{R})$ can be obtained easily by evaluation of certain determinants. In this section we show that the matrix method can be extended to other algebras which are neither semidirect products with a Heisenberg algebra nor contractions of such products. This constitutes an evidence that the procedure is not dependent on the special case of the algebras $w\mathfrak{sp}(N,\mathbb{R})$ treated, but holds for a wide class of semidirect products. Recall that the main idea in the $w\mathfrak{sp}(2N,\mathbb{R})$ case is the existence of a copy of the Levi part $\mathfrak{sp}(2N,\mathbb{R})$ in the enveloping algebra of $w\mathfrak{sp}(N,\mathbb{R})$, while in the second we used the fact that $I\mathfrak{sp}(2N,\mathbb{R})$ is obtained as a direct summand of a certain Inönü–Wigner contraction of $w\mathfrak{sp}(N,\mathbb{R})$.

Now let us consider the kinematical algebras in (3 + 1) dimensions [26]. Over the generators $\{J_i, K_i, P_i, H\}_{1\leqslant i\leqslant 3}$ the nonzero brackets of the Poincaré Lie algebra $I\mathfrak{so}(3,1)$ are given by:

$$\begin{aligned} &[J_i, J_j] = \varepsilon_{ijk}J_k; && [J_i, P_j] = \varepsilon_{ijk}P_k; && [J_i, K_j] = \varepsilon_{ijk}K_k; \\ &[H, K_i] = P_i; && [K_i, K_j] = -\varepsilon_{ijk}J_k; && [P_i, K_i] = H. \end{aligned} \tag{84}$$

In particular, $\{J_i, K_i\}$ generate the Lorentz algebra, whose Casimir operators are easily obtained using the Gel'fand method [7, 27]. If

$$\widehat{A} = \begin{pmatrix} 0 & j_3 & j_2 & -k_1 \\ -j_3 & 0 & j_1 & k_2 \\ -j_2 & -j_1 & 0 & -k_3 \\ -k_1 & k_2 & -k_3 & 0 \end{pmatrix}, \tag{85}$$

then

$$|\widehat{A} - T\,\mathrm{I\,d}_4| = T^4 + (j^\alpha j_\alpha - k^\alpha k_\alpha)T^2 - (j^\alpha k_\alpha)^2. \tag{86}$$

Since the direct sum of the Poincaré algebra and $\mathbb{R}$ cannot be obtained from an eleven-dimensional perfect Lie algebra with Heisenberg radical, the method of the enveloping algebra combined with contractions is not applicable to the inhomogeneous Lorentz group. However, the Casimir operators can still be obtained directly using the preceding matrix method. Considering the matrix

$$D = \begin{pmatrix} 0 & j_3 & j_2 & -k_1 & p_1T \\ -j_3 & 0 & j_1 & k_2 & -p_2T \\ -j_2 & -j_1 & 0 & -k_3 & p_3T \\ -k_1 & k_2 & -k_3 & 0 & hT \\ p_1 & -p_2 & p_3 & -h & 0 \end{pmatrix} \tag{87}$$

and the determinant $|D - T\,\mathrm{I\,d}_5|$, it can easily be verified that the polynomial

$$\begin{aligned} P = |D - T\,\mathrm{I\,d}_5| - T|\widehat{A} - T\,\mathrm{I\,d}_4| &= T^4(h^2 - p_\alpha p^\alpha) + \{p_\alpha p^\alpha(k_\beta k^\beta + k_\gamma k^\gamma - j_\alpha j^\alpha)\}T \\ &+ T\left\{-2(j_\alpha p^a j_\beta p^\beta + p_\alpha k^\alpha p_\beta k^\beta) + h^2 j_\alpha j^\alpha + 2h \begin{vmatrix} k_1 & k_2 & k_3 \\ p_1 & p_2 & p_3 \\ j_1 & j_2 & j_3 \end{vmatrix}\right\}, \end{aligned} \tag{88}$$

where $\alpha \neq \beta \neq \gamma$, allows us to recover the familiar Casimir operators m^2 and $W^2 = W_\mu W^\mu$ determining the mass and spin of a particle, where W_μ is the Pauli–Lubanski spin operator.

Again, we have that the matrix D decomposes as

$$D = \begin{pmatrix} 0 & j_3 & j_2 & -k_1 & p_1 \\ -j_3 & 0 & j_1 & k_2 & -p_2 \\ -j_2 & -j_1 & 0 & -k_3 & p_3 \\ -k_1 & k_2 & -k_3 & 0 & h \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & T \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ p_1 & -p_2 & p_3 & -h & 0 \end{pmatrix}, \tag{89}$$

where the first matrix on the right-hand side defines a faithful representation of the Poincaré algebra. This example shows that the matrix method can be applied to a more wide class of algebras, and does not constitute merely a reformulation of the method developed in [12] for the invariants of semidirect products of simple and Heisenberg Lie algebras. Obviously equation (45) could also be used to determine the invariants of the Galilei algebra $G(2)$ via the contraction

$$I\mathfrak{so}(3,1) \rightsquigarrow G(2) \tag{90}$$

determined by the change of basis

$$K_i = \frac{1}{\sqrt{t}} K_i, \qquad P_i = \frac{P_i}{\sqrt{t}}, \qquad i = 1, \ldots, 3. \tag{91}$$

However, the limiting procedure can again be avoided, and the Casimir operators result from the following determinants:

$$P(k_\alpha, p_\beta) := \begin{vmatrix} -T & 0 & 0 & -k_1 & p_1 T \\ 0 & -T & 0 & k_2 & -p_2 T \\ 0 & 0 & -T & -k_3 & p_3 T \\ -k_1 & k_2 & -k_3 & -T & hT \\ p_1 & -p_2 & p_3 & -h & -T \end{vmatrix} - \begin{vmatrix} -T & 0 & 0 & -k_1 \\ 0 & -T & 0 & k_2 \\ 0 & 0 & -T & -k_3 \\ -k_1 & k_2 & -k_3 & -T \end{vmatrix}$$
$$= -T^2(T^2 p_\alpha p^\alpha + 2 p_\alpha k^\alpha p_\beta k^\beta + k_\alpha k^a p_\beta p^\beta). \tag{92}$$

The independence from the variables j_α spanning the $\mathfrak{so}(3)$-Levi part follows at once from the space isotropy [9]. This implies moreover that a matrix decomposition similar to that of (48) is not associated with a faithful representation of the Galilei algebra, due to the absence of the variables corresponding to the rotation generators J_α.

7. Applications to the missing label problem

As is known, irreducible representations of a semisimple Lie algebra are labelled unambigously by the eigenvalues of Casimir operators. In a more general frame, irreducible representations of a Lie algebra $\mathfrak{g}$ are labelled using the eigenvalues of its generalized Casimir invariants [28]. By a classical result due to Racah [29] and formula (6), the number of internal labels needed equals

$$i = \tfrac{1}{2}(\dim \mathfrak{g} - \mathcal{N}(\mathfrak{g})). \tag{93}$$

When a subalgebra $\mathfrak{h}$ is used to label the basis states of $\mathfrak{g}$, it provides $\frac{1}{2}(\dim \mathfrak{h} + \mathcal{N}(\mathfrak{h}) + l'$ labels, where l' is the number of invariants of $\mathfrak{g}$ that depend only on variables of the subalgebra $\mathfrak{h}$ [28]. In order to label irreducible representations of $\mathfrak{g}$ uniquely, it is therefore necessary to find

$$n = \tfrac{1}{2}(\dim \mathfrak{g} - \mathcal{N}(\mathfrak{g}) - \dim \mathfrak{h} - \mathcal{N}(\mathfrak{h})) + l' \tag{94}$$

additional operators, which are usually called missing label operators. These are traditionally found by integrating the equations of system (4) corresponding to the subalgebra generators. The total number of available operators of this kind is easily shown to be $m = 2n$.

The matrix method developed here can not only be used to analyse different inhomogeneous algebras and their contractions, but also has potential interest in the analysis of the missing label problem [30]. To this extent, consider the symplectic algebra $\mathfrak{sp}(4,\mathbb{R})$ generated by the operators $\{a_i^\dagger a_j, a_i^\dagger a_j^\dagger, a_i a_j\}$ for $i, j = 1, 2$ and the subalgebra $\mathfrak{sl}(2,\mathbb{R})$ generated by $\{a_1^\dagger a_1, a_1^\dagger a_1^\dagger, a_1 a_1\}$. According to (94), the number of missing labels for the chain $\mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{sp}(4,\mathbb{R})$ is given by

$$n = \tfrac{1}{2}(\dim \mathfrak{sp}(4,\mathbb{R}) - \mathcal{N}(\mathfrak{sp}(4,\mathbb{R})) - \dim \mathfrak{sl}(2,\mathbb{R}) - \mathcal{N}(\mathfrak{sl}(2,\mathbb{R}))) = 2, \tag{95}$$

there are thus four available missing label operators. Using the basis (9)–(11), these operators are obtained from the subsystem formed by the following equations:

$$2x_{-1,1}\frac{\partial F}{\partial x_{-1,1}} - 2x_{1,-1}\frac{\partial F}{\partial x_{1,-1}} - x_{-1,2}\frac{\partial F}{\partial x_{-1,2}} - x_{1,-2}\frac{\partial F}{\partial x_{1,-2}} + x_{1,2}\frac{\partial F}{\partial x_{1,2}} - x_{2,1}\frac{\partial F}{\partial x_{2,1}} = 0 \tag{96}$$

$$-2x_{-1,1}\frac{\partial F}{\partial x_{1,1}} - 4x_{1,1}\frac{\partial F}{\partial x_{1,-1}} - 2x_{1,2}\frac{\partial F}{\partial x_{1,-2}} - 2x_{-1,2}\frac{\partial F}{\partial x_{2,1}} = 0 \tag{97}$$

$$2x_{1,-1}\frac{\partial F}{\partial x_{1,1}} + 4x_{1,1}\frac{\partial F}{\partial x_{-1,1}} + 2x_{2,1}\frac{\partial F}{\partial x_{-1,2}} + 2x_{1,-2}\frac{\partial F}{\partial x_{1,2}} = 0. \tag{98}$$

Instead of integrating it (although this case is extremely simple), we observe that the variables $x_{2,2}, x_{-2,2}$ and $x_{2,-2}$ do not appear in the differentials $\frac{\partial}{\partial x_{i,j}}$,[5] thus can be taken as solutions of (96)–(98). The fourth independent solution is obtained from the characteristic polynomial of the matrix

$$\mathcal{M} = \begin{pmatrix} x_{1,1} & x_{1,2} & -x_{-1,1} & -x_{-1,2} \\ x_{2,1} & 0 & -x_{-1,2} & 0 \\ x_{1,-1} & x_{1,-2} & -x_{1,1} & -x_{2,1} \\ x_{1,-2} & 0 & -x_{1,2} & 0 \end{pmatrix}. \tag{99}$$

Observe that $\mathcal{M}$ is nothing but the matrix A of (19), but where the entries corresponding to $x_{2,2}, x_{-2,2}$ and $x_{2,-2}$ have been replaced by zero. Now

$$|M - T\,\mathrm{Id}_4| = T^4 + \left(2(x_{1,-2}x_{-1,2} - x_{1,2}x_{2,1}) + \left(x_{1,-1}x_{-1,1} - x_{1,1}^2\right)\right)T^2 + (x_{1,-2}x_{-1,2} - x_{1,2}x_{2,1})^2. \tag{100}$$

Taking $I_1 = (x_{1,-2}x_{-1,2} - x_{1,2}x_{2,1})$, we obtain a fourth independent missing label operator. Observe further that the coefficient of T^2 is nothing but

$$\left(2(x_{1,-2}x_{-1,2} - x_{1,2}x_{2,1}) + \left(x_{1,-1}x_{-1,1} - x_{1,1}^2\right)\right) = 2I_1 + C_2, \tag{101}$$

where C_2 is the quadratic Casimir operator of the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra. A more interesting example is the chain $\mathfrak{sp}(4,\mathbb{R}) \hookrightarrow \mathfrak{sp}(6,\mathbb{R})$. In this case, there are $m = 6$ available missing

[5] This is obvious, since the operators $X_{2,2} = a_2^\dagger a_2$, $X_{-2,2} = a_2^\dagger a_2^\dagger$ and $X_{2,-2} = a_2 a_2$ generate an independent copy of $\mathfrak{sl}(2,\mathbb{R})$.

operators, three of which can be taken to be the variables $x_{3,3}$, $x_{-3,3}$ and $x_{3,-3}$. The other three operators can be obtained from the characteristic polynomial of the matrix

$$\mathcal{M}_1 = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & -x_{-1,1} & -x_{-1,2} & -x_{-1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} & -x_{-1,2} & -x_{-2,2} & -x_{-2,3} \\ x_{3,1} & x_{3,2} & 0 & -x_{-1,3} & -x_{-2,3} & 0 \\ x_{1,-1} & x_{1,-2} & x_{1,-3} & -x_{1,1} & -x_{2,1} & -x_{3,1} \\ x_{1,-2} & x_{2,-2} & x_{2,-3} & -x_{1,2} & -x_{2,2} & -x_{3,2} \\ x_{1,-3} & x_{2,-3} & 0 & -x_{1,3} & -x_{2,3} & 0 \end{pmatrix}. \tag{102}$$

We obtain

$$|\mathcal{M}_4 - T\,\mathrm{I\,d}_6| = T^6 + C_2T^4 + C_4T^2 + C_6, \tag{103}$$

where C_{2i} is a polynomial of degree $2i$ for $i = 1, 2, 3$. Further, for example,

$$C_2 = 2(x_{1,-3}x_{-1,3} + x_{2,-3}x_{-2,3} - x_{3,1}x_{1,3} - x_{2,3}x_{3,2}) + P_2, \tag{104}$$

where P_2 is the quadratic Casimir operator of $\mathfrak{sp}(4, \mathbb{R})$ over the given basis. The first nontrivial missing label operator can thus be taken as $L_1 = C_2 - P_2$. Simplifying C_4 and C_6 in an analogous manner, we obtain two other independent missing label operators of degrees four and six, respectively. In general, such operators can be obtained whenever we consider a chain $\mathfrak{k} \hookrightarrow \mathfrak{g}$ where the subsystem of (4) corresponding to the generators of $\mathfrak{k}$ is not dependent on all variables associated with the generators of the algebra $\mathfrak{g}$ and there exist polynomial solutions[6].

However, it must be observed that not all available missing label operators must arise by this technique. This can easily be illustrated. Take again $\mathfrak{sp}(4, \mathbb{R})$ and the two-dimensional subalgebra $\mathfrak{g}$ generated by $X_{1,1}$ and $X_{-1,1}$. Then the available missing label operators are $m = 6$. They are obtained from equations (96) and (97). Again the equations do not depend on $x_{2,2}$, $x_{-2,2}$ and $x_{2,-2}$, so we can again use matrix (99). We thus obtain the polynomial of (100):

$$T^4 + \big(2(x_{1,-2}x_{-1,2} - x_{1,2}x_{2,1}) + \big(x_{1,-1}x_{-1,1} - x_{1,1}^2\big)\big)T^2 + (x_{1,-2}x_{-1,2} - x_{1,2}x_{2,1})^2. \tag{105}$$

Taking $I_1 = (x_{1,-2}x_{-1,2} - x_{1,2}x_{2,1})$, we obtain an independent missing label operator. Since $\mathfrak{k}$ has no invariants, the coefficient of T^2 in (105) provides another independent solution of the system, namely $I_2 = x_{1,-1}x_{-1,1} - x_{1,1}^2$, which could be taken as the fifth missing label operator. But there is no possibility of obtaining a sixth independent operator by this method.

8. Conclusions

The main purpose has been to point out a formal matrix method to determine the Casimir operators of the Lie algebras $w\mathfrak{sp}(N, \mathbb{R})$ and $I\mathfrak{sp}(2N, \mathbb{R})$. Traditionally the invariants of the semidirect product $w\mathfrak{sp}(N, \mathbb{R})$ are computed by exhibiting a copy of its Levi part in the enveloping algebra, to which the classical formulae for $\mathfrak{sp}(2N, \mathbb{R})$-invariants is applied. The method has been refined and simplified by constructing a $(2N + 1) \times (2N + 1)$-matrix B depending on a parameter T whose determinant $|B - T\,\mathrm{I\,d}_{2N+1}|$ gives the invariants sought. This procedure has the advantage of avoiding the step involving the new generators generating the copy in the enveloping algebra, and seems more practical for obtaining explicit expressions of the invariants. Taking into account the contraction $w\mathfrak{sp}(N, \mathbb{R}) \rightsquigarrow I\mathfrak{sp}(2N, \mathbb{R}) \oplus \mathbb{R}$, we have obtained a similar matrix for the inhomogeneous symplectic algebras $I\mathfrak{sp}(2N, \mathbb{R})$. With this matrix, the corresponding Casimir operators can also be computed directly. The important

[6] Taking the chain $\mathfrak{u}(N) \hookrightarrow sp(2N, \mathbb{R})$, we get no missing label operators, since the equations associated with the generators of $\mathfrak{u}(N)$ involve all generators of the symplectic algebra.

fact is the parameter used in matrices also appears in the determinant, so that we cannot speak strictly of characteristic polynomials. This is due to the non-semisimplicity of the analysed algebras. However, for the inhomogeneous algebras $I\mathfrak{sp}(2N, \mathbb{R})$, the method has an interesting consequence, namely its relation to a faithful representation. The factorization (48) can thus be interpreted as a kind of generalization of the traditional matrix methods for the study of semisimple Lie algebras.

We have also enlarged and adapted our matrix approach to an important subalgebra of $w\mathfrak{sp}(N, \mathbb{R})$, the Schrödinger Lie algebra $\widehat{S}(N)$. Since for this algebra the Levi part is isomorphic to $\mathfrak{so}(N) \oplus \mathfrak{sl}(2, \mathbb{R})$, the formula to obtain the Casimir invariants has to be divided into steps, each of them corresponding to the semidirect product of the Heisenberg algebra $\mathfrak{h}_N$ with the subalgebras $\mathfrak{so}(N)$ and $\mathfrak{sl}(2, \mathbb{R})$ of the Levi part. It has been shown that these semidirect products determine the invariants of the Schrödinger algebra, and provide a maximal set of independent invariants. Their determination has also been reduced to the expansion of two determinants, providing an economical procedure to obtain the invariants of $\widehat{S}(N)$.

Although the method arises primarily from the analysis of semidirect products of symplectic algebras with Heisenberg algebras, it can also be applied to cases where the intrinsic procedure of [12] is no longer valid, such as the kinematical algebras in (3 + 1) dimensions. Under some circumstances we can still find a faithful representation of the Lie algebra associated with the matrix giving the invariants. However, the existence of such a representation can only be deduced when the Casimir operators are dependent on all generators of the algebra, as shown by the example with the Galilei algebra.

Finally, we have constructed explicit examples that show how to apply the matrix method to the study of the missing label problem. Although the validity of the argument depends on the structure of the subalgebra chain $\mathfrak{k} \hookrightarrow \mathfrak{g}$ and the existence of polynomial missing label operators, it provides useful results for low dimensional subalgebras $\mathfrak{k}$ or special types of immersions, like the chain $\mathfrak{sp}(2N-2, \mathbb{R}) \hookrightarrow \mathfrak{sp}(2N, \mathbb{R})$. Even if there are examples for which the total number of available missing labels cannot be obtained by this application of the matrix method, its simplicity makes it a technique worthy of consideration.

Acknowledgments

The author wishes to express his gratitude to the referees for drawing his attention to some mistakes and stimulating the discussion concerning the Schrödinger algebra, as well as for valuable suggestions that helped to improve the manuscript.

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